Pergamon
www.elsevier.com/locate/jappmathmech
PII: S0021-8928(99)00058-1
J. Appl. Maths Mechs, Vol. 63, No. 3, pp. 463–469, 1999

0021-8928/99/$—see front matter

INSTABILITY OF THE EQUILIBRIUM POSITION OF A HEAVY SPHERE IN A STEADY NON-UNIFORM POTENTIAL FLOW†

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(*Received* 28 *March* 1997)

A theorem on the instability of any equilibrium position of a heavy sphere of finite radius in an arbitrary steady potential non-uniform flow of an ideal fluid is proved. © 1999 Elsevier Science Ltd. All rights reserved.

The problem of the stability of the equilibrium position of a heavy body in a given steady non-uniform potential flow has been formulated for a small sphere [1] and a small body of arbitrary shape [2]. Lagrangians have been constructed and the fact that there are no points of a local minimum of the potential energy has been demonstrated. It has also been shown that the Lagrangian for a small sphere has no linear terms with respect to the velocity and the conclusion has been drawn that any equilibrium position is unstable [1].

Below we construct an exact expression for the Lagrangian for a sphere of finite radius which moves in a given non-uniform potential flow. The problem of the stability of the equilibrium position of the sphere in a non-uniform steady flow in a potential field of mass forces is formulated and solved. The same properties of the Lagrangian are proved for a sphere of finite radius as have been established for a small sphere [1]. Hence, from the well-known results on the inversion of the Lagrange–Dirichlet theorem, it is concluded that the sphere has no stable equilibrium.

1. THE EXACT DYNAMICAL PRINCIPLE AND THE EQUATIONS OF MOTION

Suppose we are given an arbitrary potential flow of an ideal incompressible fluid with a velocity field which depends on the time t and the coordinates $\mathbf{x}(x, y, z)$

$$\mathbf{v}_0(t,\mathbf{x}) = \nabla\Phi_0(t,\mathbf{x}),\ \ \nabla^2\Phi_0(t,\mathbf{x}) = 0 \tag{1.1}$$
$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

and the pressure $p_0(t, \mathbf{x})$. If the mass forces have a potential U, then the pressure and velocity are related by the Cauchy–Lagrange integral

$$p_0 + \rho\left(\frac{\partial\Phi_0}{\partial t} + \frac{v_0^2}{2}\right) - \rho U = f(t) \tag{1.2}$$

The motion of a body in a flow with velocity field (1.1) can be described using Hamilton's variational principle, the exact formulation of which for this case is given in [3, 4].

The real motion of the body between its two given positions is different from the kinematically possible motions in the same time interval $t \in (t_1, t_2)$ in that, for real motion, the variation of the Hamilton action is equal to zero

$$\delta\int_{t_1}^{t_2} L dt = 0,\ \ L = T_0 - \Pi_0 + \Lambda \tag{1.3}$$

where T_0 and Π_0 are the kinetic and potential energies of the body and Λ is the associated Lagrange function.

†*Prikl. Mat. Mekh.* Vol. 63, No. 3, pp. 481–488, 1999.

The following exact representation was derived in [3]

$$\Lambda = \frac{\rho}{2}\int_{\Omega}(\mathbf{v}-\mathbf{v}_0)^2 dV - \int_V p_0 dV \tag{1.4}$$

where $\mathbf{v} = \nabla\Phi$ is the potential velocity field of the fluid perturbed by the body, V is the volume of the body and Ω is the region occupied by the fluid.

If $q_1, q_2, \ldots, q_6$ are generalized coordinates of the body, then formulae (1.1)–(1.4) define the Lagrange function of time t, generalized coordinates q_α and velocities $\dot{q}_\alpha$ ($\alpha = 1, 2, \ldots, 6$). The equations of motion of the body can be written in the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\alpha} - \frac{\partial L}{\partial q_\alpha} = 0, \quad \alpha = 1, 2, \ldots, 6 \tag{1.5}$$

Below we shall call the potential velocity field of general form $\mathbf{v}_0(t, \mathbf{x})$ a non-uniform flow, and the time-independent velocity field $\mathbf{v}_0(\mathbf{x})$—a non-uniform steady flow. The coordinate-independent velocity field $\mathbf{v}_0(t)$ will be called uniform flow.

2. A SMALL SPHERE IN NON-UNIFORM FLOW. INSTABILITY OF THE EQUILIBRIUM OF A SMALL SPHERE IN A STEADY NON-UNIFORM FLOW

We can take the Cartesian coordinates of the centre of the sphere $\mathbf{x}_0(x_0, y_0, z_0)$ as generalized coordinates. For a sphere of sufficiently small radius a with potential $U = -gz$ (the gravity field) the functions (1.4) and (1.3) have the following leading terms of the asymptotic expansion

$$\begin{aligned}
&\Lambda = \frac{1}{2}\rho V|\dot{\mathbf{x}}_0 - \mathbf{v}_0(t, \mathbf{x}_0)|^2 - Vp_0(t, \mathbf{x}_0) + O(a^5)\\
&T_0 = \frac{1}{2}M\dot{\mathbf{x}}_0^2, \quad \Pi_0 = gMz_0\\
&L = \frac{1}{2}M\dot{\mathbf{x}}_0^2 + \frac{1}{4}\rho V|\dot{\mathbf{x}}_0 - \mathbf{v}_0(t, \mathbf{x}_0)|^2 - Vp_0(t, \mathbf{x}_0) - gMz_0
\end{aligned} \tag{2.1}$$

where M is the mass and $V = 4\pi a^3/3$ is the volume of the sphere.

It can be shown that the terms in $\dot{\mathbf{x}}_0$ which are linear with respect to the Lagrange function L can be eliminated. For, adding the total derivative

$$\frac{d}{dt}\left(\frac{1}{2}\rho V\Phi_0(t, \mathbf{x}_0)\right) = \frac{1}{2}\rho V\left(\frac{\partial\Phi_0}{\partial t} + \dot{\mathbf{x}}_0\mathbf{v}_0\right)$$

to the right-hand side of the last expression, we obtain

$$L = \frac{1}{2}M'\dot{\mathbf{x}}_0^2 - W, \quad W = \frac{3}{2}Vp_0(t, \mathbf{x}_0) + gz_0M', \quad M' = M + \frac{1}{2}\rho V \tag{2.2}$$

Then the equations of motion (1.5) will take the form

$$M'\ddot{\mathbf{x}}_0 = -\nabla_0 W(t, \mathbf{x}_0), \quad \nabla_0 = \mathbf{i}\frac{\partial}{\partial x_0} + \mathbf{j}\frac{\partial}{\partial y_0} + \mathbf{k}\frac{\partial}{\partial z_0} \tag{2.3}$$

The Lagrange function (2.2) and equation of motion (2.3) of a small sphere in a uniform flow were obtained in a different way, apparently for the first time, in [1]. Equations (2.3) were derived by Kelvin for a steady non-uniform flow [5]. The force acting on a fixed small sphere in a non-uniform unsteady flow was determined by Zhukovskii [6].

In a non-uniform flow a small sphere of mass M moves like a point with mass M' in a force field with potential—W. (Below we shall refer to the function—W as the force potential, and the inverse function as the potential energy.) The effect of non-uniform flow is to add to the mass of the sphere M an associated mass $\rho V/2$ and to the potential of external forces the term—$3Vp_0(t, \mathbf{x}_0)/2$.

The problem of the stability of the equilibrium of a small sphere in a steady non-uniform flow was posed in [1] and reduced to an investigation of the equilibrium point $\mathbf{x} = 0$ and its stability in accordance with the equations of motion (2.3). The equilibrium condition $\nabla_0 W = 0$ or $\nabla p_0 = 2\mathbf{g}M'V^{-1}/3$ at $\mathbf{x} = 0$ can also be satisfied by the appropriate choice of the potential of non-uniform flow $\Phi_0(\mathbf{x})$. It was noted that in steady uniform flow the law of conservation of energy

$$\frac{1}{2}M'\dot{\mathbf{x}}_0^2 + W(\mathbf{x}_0) = E$$

holds. In view of the well-known inequality $\nabla_0^2 p_0(\mathbf{x}_0) \leqslant 0$, the function $W(\mathbf{x}_0)$ is subharmonic, i.e. it satisfies the inequality

$$\nabla_0^2 W(\mathbf{x}_0) = \frac{3}{2}V\nabla_0^2 p_0(\mathbf{x}_0) \leqslant 0$$

As we know, this function cannot have a local minimum. This property of the given dynamical system [1] is the analogue of Earnshaw's theorem in electrodynamics on the lack of a stable equilibrium position of a point charge in an electrostatic field. Since the potential energy is a harmonic function, it has no maximum at any point. The instability of systems with a subharmonic force function is discussed in Section 5.

We shall prove that all the properties determining the instability of a small sphere also hold for a sphere of finite radius. Thus in Section 3 we shall prove that the Lagrangian of the system for general steady non-uniform flow can be represented in the form of the sum $L = L_2 + L_0$, where L_2 is a positive-definite form which is quadratic with respect to the velocities and L_0 is the force potential, which is independent of the velocities. Then in Section 4 we establish that $\nabla_0^2 L_0 > 0$ is a subharmonic force potential (Section 5) we deduce a theorem on the instability of any equilibrium position of a sphere of finite radius in a steady non-uniform flow.

3. PROOF THAT THERE ARE NO GYROSCOPIC FORCES

We shall show that the linear terms in $\dot{\mathbf{x}}_0$ in the function Λ (1.4) can be eliminated not only in the case of a small sphere, but also for a sphere of finite radius which moves in a steady non-uniform flow with potential $\Phi_0(\mathbf{x}_0)$. To do so, we shall represent the velocity field in the first integral of (1.4) in the form

$$\mathbf{v} - \mathbf{v}_0 = \nabla(\varphi - \tilde{\varphi}) \tag{3.1}$$

Both functions φ and $\tilde{\varphi}$ are harmonic, tend to zero at infinity and on the boundary of the sphere $\partial V(\mathbf{x}_0)$ satisfy the conditions

$$\frac{\partial\varphi}{\partial n} = \dot{\mathbf{x}}_0\mathbf{n}, \quad \frac{\partial\tilde{\varphi}}{\partial n} = \frac{\partial\Phi_0}{\partial n}; \quad \mathbf{n} = (x - x_0, \ y - y_0, \ z - z_0)/a \tag{3.2}$$

where $\mathbf{n}$ is the vector of the unit normal to the sphere surface ∂V.

Thus, the functions φ and $\tilde{\varphi}$ satisfy the equation and boundary conditions of outer Neumann problems, which have unique solutions. The solutions for φ and $\tilde{\varphi}$ can be written explicitly in the form

$$\varphi(\mathbf{x}, \mathbf{x}_0) = -\frac{a^3}{2r^2}(\dot{\mathbf{x}}_0\mathbf{n}) \tag{3.3}$$

$$\tilde{\varphi}(\mathbf{x}, \mathbf{x}_0) = -\frac{1}{a}\int_0^{a^2/r} R\frac{\partial\Phi_0(\mathbf{x}_0 + R\mathbf{n})}{\partial R}dR \tag{3.4}$$

$$r^2 = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2$$

$$R^2 = (x' - x_0)^2 + (y' - y_0)^2 + (z' - z_0)^2$$

The potential φ of (3.3) defines the well-known flow about a sphere moving in a fluid which is at rest at infinity. The potential $\tilde{\varphi}$ of (3.4) is the potential of the part of the flow perturbed by the sphere in a non-uniform flow with potential Φ_0 and is found from Weiss's theorem [7, p. 467]. We shall refer to the function $\tilde{\varphi}$ as the Weiss potential. We substitute expression (3.1) into (1.4) and use Green's formula, noting that the function φ is linear in the velocity and $\tilde{\varphi}$ is independent of the velocity $\dot{\mathbf{x}}_0$. We separate out the term Λ_2, which is quadratic, the term Λ_1 which is linear and the term $\dot{\mathbf{x}}_0$ which is independent of Λ_0, and obtain

$$\Lambda = -\frac{1}{2}\rho \int\limits_{\partial V(\mathbf{x}_0)} (\varphi - \tilde{\varphi})\left(\frac{\partial\varphi}{\partial n} - \frac{\partial\tilde{\varphi}}{\partial n}\right) dS - \int\limits_{V(\mathbf{x}_0)} p_0 dV = \Lambda_0 + \Lambda_2$$

$$\Lambda_0 = -\frac{1}{2}\rho \int\limits_{\partial V(\mathbf{x}_0)} \tilde{\varphi}\frac{\partial\tilde{\varphi}}{\partial n} dS - \int\limits_{V(\mathbf{x}_0)} p_0 dV, \quad \Lambda_2 = \frac{1}{4}\rho V(\dot{x}_0^2 + \dot{y}_0^2 + \dot{z}_0^2) \tag{3.5}$$

where $\mathbf{n}$ is the outward normal to the sphere V.

We have omitted the term

$$\Lambda_1 = \frac{\rho}{2} \int\limits_{\partial V(\mathbf{x}_0)} \left(\varphi\frac{\partial\tilde{\varphi}}{\partial n} + \tilde{\varphi}\frac{\partial\varphi}{\partial n}\right) dS \tag{3.6}$$

since it is a total derivative with respect to time

$$\Lambda_1 = \frac{d}{dt}\left(-\frac{1}{2}\rho V \Phi_0(x_0, y_0, z_0)\right) \tag{3.7}$$

and makes no contribution to the equation of the sphere motion.

In fact, by Green's formula, using boundary condition (3.2), we can transform (3.6) to the form

$$\Lambda_1 = \rho \int\limits_{\partial V(\mathbf{x}_0)} \varphi \frac{\partial \Phi_0}{\partial n} dS$$

On the boundary $\partial V(\mathbf{x}_0)$ the function φ is equal to $P_1 = -(\dot{x}_0\Delta x + \dot{y}_0\Delta y + \dot{z}_0\Delta z)/2$. The function $P_1\nabla\Phi_0$ is defined and differentiable inside V and Gauss's theorem can be applied to the last integral. Then, using the property of the harmonic function $\nabla(P_1\nabla\Phi_0)$, we obtain

$$\Lambda_1 = \rho \int\limits_{V(\mathbf{x}_0)} \nabla(P_1\nabla\Phi_0) dV = -\frac{1}{2}\rho V\left(\dot{x}_0\frac{\partial\Phi_0}{\partial x_0} + \dot{y}_0\frac{\partial\Phi_0}{\partial y_0} + \dot{z}_0\frac{\partial\Phi_0}{\partial z_0}\right)$$

From the theorem on differentiating a composite function, the last expression is the same as (3.7), as was to be proved.

Thus, the exact expression of the Lagrangian L can be represented by two terms: the positive-definite form L_2, which is quadratic in the velocities, and the force function L_0, which is independent of the body velocities, and these are found using (1.3) and (3.5)

$$L = L_2 + L_0$$

$$L_2(\theta, \theta_1, \theta_2, \dot{\theta}, \dot{\theta}_1, \dot{\theta}_2, \dot{\mathbf{x}}_0) = \rho |\dot{\mathbf{x}}_0|^2 + T_0 \tag{3.8}$$

$$L_0(\theta, \mathbf{x}_0) = -gM(z_0 + l\cos\theta) + \frac{1}{2}\rho \int\limits_{\Omega(\mathbf{x}_0)} \tilde{\mathbf{v}}^2(\mathbf{x}, \mathbf{x}_0) dV - \int\limits_{V(\mathbf{x}_0)} p_0(\mathbf{x}) dV \tag{3.9}$$

The first integral in L_0 is transformed using Green's formula, and the vector $\tilde{\mathbf{v}} = \nabla\tilde{\varphi}$ is the velocity with Weiss's potential (3.4), which depends on the actual coordinates $\mathbf{x}$ and the coordinates of the centre of the sphere $\mathbf{x}_0$.

The generalized coordinates are taken as the three coordinates of the centre of the sphere $O(x_0, y_0, z_0)$ and the Euler angles θ, θ_1, θ_2, where θ is the angle between the axis Oz directed vertically upwards and the vector $\mathbf{l}(x_c - x_0, y_c - y_0, z_c - z_0)$, and the point $C(x_c, y_c, z_c)$ is the centre of mass of the sphere (see Fig. 1). The kinetic energy T_0 of a solid sphere comprises the kinetic energy of the motion of the centre of mass and a the kinetic energy of motion about the centre of mass (König's theorem)

$$T_0(\theta, \theta_1, \theta_2, \dot{\theta}, \dot{\theta}_1, \dot{\theta}_2, \dot{\mathbf{x}}_0) = \frac{1}{2}M(\dot{\mathbf{x}}_0^2 + (\omega \times \mathbf{l})^2) + \frac{1}{2}\sum_{1 \le i,j \le 3} I_{ij}\omega_i\omega_j$$

where $\omega = (\omega_1, \omega_2, \omega_3)$ is the angular velocity vector and I_{ij} are the moments of inertia of the solid $(i, j = 1, 2, 3)$.

We have thus obtained the required representation of the Lagrange function.

4. PROOF OF THE FACT THAT THE FORCE POTENTIAL IS SUBHARMONIC

The Lagrange function (3.8) corresponds to a conservation system with six degrees of freedom. We will prove that the force function L_2 is subharmonic: $\nabla^2 L_0(\mathbf{x}_0) \geqslant 0$. We need the following three lemmas.

Lemma 1. We have the following differentiation formulae

$$\nabla_0 \int\limits_{\Omega(\mathbf{x}_0)} \tilde{F}(\mathbf{x}, \mathbf{x}_0) dV = \int\limits_{\Omega(\mathbf{x}_0)} (\nabla + \nabla_0) \tilde{F}(\mathbf{x}, \mathbf{x}_0) dV \tag{4.1}$$

$$\nabla_0 \int\limits_{V(\mathbf{x}_0)} F(\mathbf{x}, \mathbf{x}_0) dV = \int\limits_{V(\mathbf{x}_0)} (\nabla + \nabla_0) F(\mathbf{x}, \mathbf{x}_0) dV \tag{4.2}$$

where $V(\mathbf{x}_0)$ is the interior of a sphere of radius a with centre at the point $\mathbf{x}_0$ and $\Omega(\mathbf{x}_0)$ is its exterior. The functions $\tilde{F}$ and F depend on the variables $\mathbf{x}(x, y, z)$, $\mathbf{x}_0(x_0, y_0, z_0)$ and are differentiable in the respective regions $\Omega(\mathbf{x}_0)$, $V(\mathbf{x}_0)$ and the integrals in (4.1) are assumed to be absolutely convergent.

The identity (4.1) is proved by finding the derivative. For the increment of the integral corresponding to the increment of the argument $\Delta\mathbf{w}_0$, we have

$$\int\limits_{\Omega(x_0+\Delta x)} \tilde{F}(\mathbf{x}, \mathbf{x}_0 + \Delta\mathbf{x}) dV - \int\limits_{\Omega(x_0)} \tilde{F}(\mathbf{x}, \mathbf{x}_0) dV =$$

$$= \left(\int\limits_{\Omega(x_0)} \nabla_0 F(\mathbf{x}, \mathbf{x}_0) dV + \int\limits_{\partial\Omega(x_0)} F(\mathbf{x}, \mathbf{x}_0) \mathbf{n} dS \right) \Delta\mathbf{x} + O(\Delta\mathbf{x})^2$$

where $\Omega(\mathbf{x}_0 + \Delta\mathbf{x})$ is the region exterior to a sphere with displaced centre at the point $\mathbf{x}_0 + \Delta\mathbf{x}$), $\Omega(\mathbf{x}_0)$ is the region exterior to a sphere with centre at $\mathbf{x}_0$ and $\mathbf{n}$ is the inward normal to V and the outward normal to $\Omega(\mathbf{x}_0)$. Using Gauss's formula and dividing by the increment of the argument, we obtain the required identity (4.1). The identity (4.2) is derived in the same way.

Lemma 2. For the Weiss potential $\varphi(\mathbf{x}, \mathbf{x}_0)$ defined by (3.4) and the potential of a non-uniform flow $\Phi_0(\mathbf{x})$, we have the identities

$$(\nabla + \nabla_0)(\nabla + \nabla_0)\tilde{\varphi}(\mathbf{x}, \mathbf{x}_0) = 0 \tag{4.3}$$

We will prove this using Eq. (3.4). Let the arguments $\mathbf{x}$ and $\mathbf{x}_0$ be given the same small increment

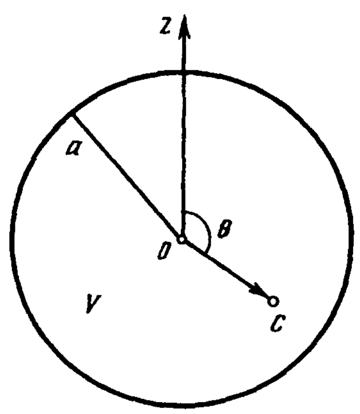

Fig. 1.

$\Delta\mathbf{x}$. Then the only thing that changes in (3.4) is that $\mathbf{x}_0$ becomes $\mathbf{x}_0 + \Delta\mathbf{x}$, while the other variables r, $\mathbf{n}$ and R remain unchanged. The corresponding increment of the function $\tilde{\varphi}$ satisfies the equation

$$\Delta\mathbf{x}(\nabla + \nabla_0)\tilde{\varphi} = -\frac{\Delta\mathbf{x}}{a}\int_0^{a^2/r} R\frac{\partial}{\partial R}\nabla_0\Phi_0(\mathbf{x}_0 + R\mathbf{n})dR$$

Hence

$$(\nabla + \nabla_0)\tilde{\varphi} = -\frac{1}{a}\int_0^{a^2/r} R\frac{\partial}{\partial R}\nabla_0\Phi_0(\mathbf{x}_0 + R\mathbf{n})dR$$

Applying the operator $(\nabla + \nabla_0)$ twice to this identity, by a similar argument we obtain Eq. (4.3) (using the fact that the function $\Phi_0(\mathbf{x})$ is harmonic).

Lemma 3. The square of the velocity $\tilde{\mathbf{v}}(\mathbf{x}, \mathbf{x}_0) = \nabla\tilde{\varphi}$ with Weiss potential and the pressure $p_0(\mathbf{x})$, determined from the Cauchy–Lagrange integral (1.2), satisfy the inequalities

$$(\nabla + \nabla_0)(\nabla + \nabla_0)\tilde{\mathbf{v}}^2(\mathbf{x}, \mathbf{x}_0) = 2\left(\left|(\nabla + \nabla_0)\frac{\partial\tilde{\varphi}}{\partial x}\right|^2 + \left|(\nabla + \nabla_0)\frac{\partial\tilde{\varphi}}{\partial y}\right|^2 + \left|(\nabla + \nabla_0)\frac{\partial\tilde{\varphi}}{\partial z}\right|^2\right) \geqslant 0 \tag{4.4}$$

$$-(\nabla + \nabla_0)(\nabla + \nabla_0)p_0(\mathbf{x}) = \rho\left(\left|\nabla\frac{\partial\Phi_0}{\partial x}\right|^2 + \left|\nabla\frac{\partial\Phi_0}{\partial y}\right|^2 + \left|\nabla\frac{\partial\Phi_0}{\partial z}\right|^2\right) \geqslant 0$$

Proof. Bearing in mind Eq. (4.3) and the consequent identity and making the obvious transformation, we obtain the first inequality of (4.4)

$$(\nabla + \nabla_0)(\nabla + \nabla_0)\tilde{\mathbf{v}}(\mathbf{x}, \mathbf{x}_0) = \nabla((\nabla + \nabla_0)(\nabla + \nabla_0)\tilde{\varphi}(\mathbf{x}, \mathbf{x}_0)) = 0$$

$$(\nabla + \nabla_0)(\nabla + \nabla_0)\tilde{\mathbf{v}}^2(\mathbf{x}) = 2\sum_{k=1}^{3}(\tilde{v}_k(\nabla + \nabla_0)(\nabla + \nabla_0)\tilde{v}_k + |(\nabla + \nabla_0)\tilde{v}_k|^2) =$$

$$= \sum_{k=1}^{3} 2|(\nabla + \nabla_0)\tilde{v}_k|^2$$

The second inequality of (4.4) is derived in the same way.

We now prove that the force function $L_0(\mathbf{x}_0)$ is subharmonic.

We apply the operator ∇_0 twice to the function $L_0(\mathbf{x}_0)$ in (3.9). Using identities (4.1) and (4.2) in succession we obtain

$$\nabla_0^2 L_0(\mathbf{x}_0) = \frac{\rho}{2}\int_{\Omega(\mathbf{x}_0)}(\nabla + \nabla_0)(\nabla + \nabla_0)\tilde{\mathbf{v}}^2 dV - \int_{V(\mathbf{x}_0)}\nabla^2 p_0(\mathbf{x})dV \tag{4.5}$$

Applying inequalities (4.4) of Lemma 3 to the integrands in (4.5) we obtain the required condition

$$\nabla^2 L_0(\mathbf{x}_0) \geqslant 0 \tag{4.6}$$

from which it follows that the function $L_0(\mathbf{x}_0)$ cannot have a local maximum at any point. Hence, using well-known results on the inversion of the Lagrange–Dirichlet theorem we obtain a theorem on the stability of the equilibrium of a sphere of finite radius a in a non-uniform steady flow (Section 5).

We can prove a more general statement. The angles θ_1, θ_2 may be cyclic coordinates. Then the sphere can perform steady motion with constant angular momentum with fixed centre $\mathbf{x}_0$. Any such steady motion will also be unstable. For, the function T_0 does not depend explicitly on the coordinates $\mathbf{x}_0$. Thus by using Routh's method to eliminate the cyclic coordinates, we find that the additional potential energy will be independent of $\mathbf{x}_0$ and all the results on the lack of a local minimum of the potential energy remain valid.

5. THEOREMS ON THE STABILITY OF EQUILIBRIUM IN THE CASE OF A SUBHARMONIC FORCE POTENTIAL

Suppose that at the origin of coordinates the system with Lagrangian (3.8) satisfied the equilibrium condition

$$\nabla_0 L_0(\theta, \mathbf{x}_0) = \mathbf{g}M + \frac{1}{2}\rho\nabla_0 \int\limits_{\Omega(\mathbf{x}_0)} \tilde{\mathbf{v}}^2(\mathbf{x}, \mathbf{x}_0)dV - \nabla_0 \int\limits_{V(\mathbf{x}_0)} p_0(\mathbf{x})dV = 0 \tag{5.1}$$

where the velocity $\tilde{\mathbf{v}}(\mathbf{x}, \mathbf{x}_0)$ with Weiss potential and pressure of non-uniform flow $p_0(\mathbf{x})$ can be expressed in terms of the arbitrary function $\Phi_0(\mathbf{x})$ using formulae (3.4) and (1.2). The arbitrary function $\Phi_0(\mathbf{x})$ can be chosen to satisfy condition (5.1). For example, we could choose a family of potentials $K\Phi_0$ with axial symmetry with respect to the z axis. Then both integrals in (5.1) will be in the direction of the gravitational acceleration $\mathbf{g}$ and proportional to K_2. Equation (5.1) will be satisfied with the appropriate value of the coefficient K.

The expansion of the force potential in a Maclaurin series at the equilibrium point $\mathbf{x} = 0$ will have the form

$$L_0 = L^{(2)} + L^{(3)} + L^{(4)} + \dots \tag{5.2}$$

where $L^{(n)}$ is a polynomial of degree n, homogeneous in the coordinates.

If $L^{(2)} \neq 0$, by virtue of (4.6) $\nabla^2 L^{(2)} \geqslant 0$. In that case the instability of the equilibrium follows from Lyapunov's theorem on instability in a first approximation (see [8], for example). This is the most common case of the equilibrium of such a system. Note that Earnshaw concluded, on the basis of this result, that a charge is unstable in any electrostatic field.

In the degenerate case $L^{(2)} = 0$ series (5.2) can begin with a homogeneous polynomial of higher than the second degree

$$L_0 = L^{(n)} + L^{(n+1)} + \dots, \quad n > 2$$

and, from subharmonic condition (4.6), $L^{(n)}$ will not have a local maximum at zero. The theorems of Chetayev [8] and others [9, pp. 88–90] on the inversion of the Lagrange–Dirichlet theorem, are used to prove that the equilibrium is unstable under special conditions in this degenerate case. The result obtained in [10] completely solves the problem of proving the theorem on the instability of equilibrium of the given system: "Let the force function be subharmonic and let its Maclaurin series be non-zero; then equilibrium $\mathbf{x} = 0$ is unstable". Instability in the Earnshaw problem in the degenerate case follows from this result in [10].

I wish to thank F. L. Chernous'ko for his interest.

This research was supported financially by the Russian Foundation for Basic Research (96-01-00399).

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Translated by R.L.